

A NOTE ON LOCAL HARDY SPACES

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ABSTRACT. We consider a non-negative self-adjoint operator L satisfying generalized Gaussian estimates on a doubling metric measure space, and show that if L has a spectral gap then the local and global Hardy spaces defined by means of appropriate square functions coincide.

1. INTRODUCTION

The real-variable Hardy space H^1 was introduced by E. M. Stein and G. Weiss [18, 19] as a harmonic analytic substitute for the endpoint Lebesgue space \mathcal{L}^1 . A larger local Hardy space \mathfrak{h}^1 , which is better suited for smooth Fourier multipliers, was studied by D. Goldberg [11].

The classical setting is based on the Euclidean space \mathbb{R}^n and arises, somewhat implicitly, from its Laplacian $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. The theory has since then been extended into various directions: to second order elliptic operators [14, 15] and Schrödinger operators [9] on the Euclidean space; to the first order framework of Hodge–Dirac operators on Riemannian manifolds with doubling volume property [2] and to the corresponding local setting [6]; to non-negative self-adjoint operators satisfying Davies–Gaffney estimates of order 2 on doubling metric measure spaces [13] and to operators with estimates of higher order [17, 20, 10]. Local Hardy spaces for operators with pointwise Gaussian upper bounds have also been considered in [16, 12].

The main result of this article concerns non-negative self-adjoint operators L that have a spectral gap in the sense that $\inf \sigma(L) =: \lambda_0 > 0$. In such a case, the size of any function is captured by its *local* square function, which only takes into account short time diffusion of the semigroup $(e^{-tL})_{t>0}$. Indeed, denoting by E the spectral measure of L , we have

$$\frac{1}{4}\|f\|_2^2 = \left\| \left(\int_0^1 |tLe^{-tL}f|^2 \frac{dt}{t} \right)^{1/2} \right\|_2^2 + \int_{\lambda_0}^\infty \left(\frac{\lambda}{2} + \frac{1}{4} \right) e^{-2\lambda} dE_{f,f}(\lambda),$$

and therefore

$$\left\| \left(\int_0^1 |tLe^{-tL}f|^2 \frac{dt}{t} \right)^{1/2} \right\|_2 \geq c\|f\|_2,$$

where $c = (\frac{1}{4} - (\frac{\lambda_0}{2} + \frac{1}{4})e^{-2\lambda_0})^{1/2} > 0$. In other words, the global square function is controlled by its local version.

Assume that (M, d, μ) is a doubling metric measure space: there exists a number $n > 0$ such that for every ball $B \subset M$ we have

$$\mu(\alpha B) \lesssim \alpha^n \mu(B),$$

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whenever $\alpha \geq 1$. Assume moreover that L is a non-negative self-adjoint operator of ‘order $m \geq 2$ ’ on the Lebesgue space \mathcal{L}^2 (see Definition 1).

The Hardy space H_L^1 is defined by means of the conical square function

$$Sf(x) = \left(\int_0^\infty \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |t^m L e^{-t^m L} f(y)|^2 d\mu(y) \frac{dt}{t} \right)^{1/2}, \quad x \in M,$$

as a completion with respect to the norm $\|f\|_{H_L^1} = \|Sf\|_1$. Similarly, the local Hardy space \mathfrak{h}_L^1 is defined by means of S_{loc} in which \int_0^∞ is replaced by \int_0^1 and the norm is defined as $\|f\|_{\mathfrak{h}_L^1} = \|S_{loc}f\|_1 + \|f\|_1$. From $\|Sf\|_1 \gtrsim \|f\|_1$ it is immediate that $H_L^1 \subset \mathfrak{h}_L^1$. For the purposes of this article we may think of these as incomplete subspaces of \mathcal{L}^2 in order to avoid technical complications.

The main result (Theorem 7) says that if $(e^{-tL})_{t>0}$ satisfies generalized Gaussian (1, 2)-estimates (see Definition 1) and L has a spectral gap, then actually $\mathfrak{h}_L^1 = H_L^1$.

A prototypical example of a second order operator to which the result applies is the Schrödinger operator $L = -\Delta + |x|^2$ on \mathbb{R}^n . Moreover, by [8, Theorem 3.1], the heat kernel of any generalized Schrödinger operator $L = (-\Delta)^{m/2} + V$ with non-negative locally integrable potential V on \mathbb{R}^n satisfies pointwise Gaussian upper bounds of order m (and therefore also generalized Gaussian (1, 2)-estimates) whenever m is an even integer greater than n .

Notation. We denote by $D(L)$ and $R(L)$ the domain and range in \mathcal{L}^2 of the linear operator L . The notation $\|\cdot\|_{p \rightarrow 2}$ stands for the operator norm from \mathcal{L}^p to \mathcal{L}^2 . The radius of a ball $B \subset M$ is denoted by r_B .

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2. OFF-DIAGONAL ESTIMATES

In this section we discuss the generalized Gaussian estimates, which in this form and generality were introduced by S. Blunck and P. C. Kunstmann in [3, 5, 4].

Definition 1. Let $m \geq 2$ and $p \in [1, 2]$. A family $(T_t)_{t>0}$ of linear operators on \mathcal{L}^2 is said to satisfy *generalized Gaussian (p, 2)-estimates (of order m)* (abbreviated $\text{GGE}_m(p, 2)$) if there exists a constant c such that for all $x, y \in M$ and all $t > 0$ we have

$$\|1_{B(x,t)} T_t^m 1_{B(y,t)}\|_{p \rightarrow 2} \lesssim \mu(B(x,t))^{-(\frac{1}{p}-\frac{1}{2})} \exp\left(-c\left(\frac{d(x,y)}{t}\right)^{\frac{m}{m-1}}\right).$$

For a ball $B \subset M$ we write $C_k(B) = 2^k B \setminus 2^{k-1} B$, when $k \geq 1$, $C_0(B) = B$, and $C_k^*(B) = 2^{k+1} B \setminus 2^{k-2} B$, $C_1^*(B) = 4B$, $C_0^*(B) = 2B$.

The following result collects the required off-diagonal estimates. A more systematic treatment can be found for instance in M. Uhl’s thesis [20].

Proposition 2. *If $(T_t)_{t>0}$ satisfies $\text{GGE}_m(p, 2)$ with $1 \leq p \leq 2$, then for every ball $B \subset M$ and every $k \geq 1$ we have*

$$\|1_{C_k(B)} T_t^m 1_B\|_{p \rightarrow 2} \lesssim \mu(B)^{-n(\frac{1}{p}-\frac{1}{2})} \left(1 + \frac{r_B}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})} 2^{nk} \exp\left(-c\left(\frac{2^k r_B}{t}\right)^{\frac{m}{m-1}}\right), \quad t > 0.$$

Moreover, if $(T_t)_{t>0}$ satisfies $\text{GGE}_m(2, 2)$, then for every ball $B \subset M$ and every $k \geq 0$ we have

$$\|1_{C_k(B)} T_t^m 1_{M \setminus C_k^*(B)}\|_{2 \rightarrow 2} \lesssim \left(\frac{t}{2^k r_B}\right)^{n+2}, \quad t > 0.$$

Proof. Assume that $(T_t)_{t>0}$ satisfies $\text{GGE}_m(p, 2)$ with $1 \leq p \leq 2$ and consider a ball $B \subset M$. By [20, Lemma 2.6 b)] we have for all $j \geq 2$ and $t > 0$,

$$(1) \quad \|1_{(j+1)B \setminus jB} T_{t^m} 1_B\|_{p \rightarrow 2} \lesssim \mu(B)^{-n(\frac{1}{p}-\frac{1}{2})} \left(1 + \frac{r_B}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})} j^n \exp\left(-c\left(\frac{jr_B}{t}\right)^{\frac{m}{m-1}}\right).$$

Let $k \geq 1$. Writing

$$C_k(B) = \bigcup_{j=2^{k-1}}^{2^k-1} (j+1)B \setminus jB,$$

and noticing that

$$\sum_{j=2^{k-1}}^{2^k-1} j^n \lesssim 2^{nk},$$

we obtain from (1) that for all $t > 0$,

$$(2) \quad \|1_{C_k(B)} T_{t^m} 1_B\|_{p \rightarrow 2} \lesssim \mu(B)^{-n(\frac{1}{p}-\frac{1}{2})} \left(1 + \frac{r_B}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})} 2^{nk} \exp\left(-c\left(\frac{2^k r_B}{t}\right)^{\frac{m}{m-1}}\right),$$

which proves the first case.

For the second case we assume that $(T_t)_{t>0}$ satisfies $\text{GGE}_m(2, 2)$. Writing, for any ball $B \subset M$,

$$M \setminus 2B = \bigcup_{j=2}^{\infty} (j+1)B \setminus jB$$

and making use of the estimate

$$(3) \quad \exp(-c\alpha^{\frac{m}{m-1}}) \lesssim \alpha^{-n-2} \exp(-c'\alpha^{\frac{m}{m-1}})$$

we see that for all $t > 0$

$$\begin{aligned} \sum_{j=2}^{\infty} j^n \exp\left(-c\left(\frac{jr_B}{t}\right)^{\frac{m}{m-1}}\right) &\leq \sum_{j=2}^{\infty} \frac{t^n}{r_B^n} \left(\frac{jr_B}{t}\right)^n \exp\left(-c\left(\frac{jr_B}{t}\right)^{\frac{m}{m-1}}\right) \\ &\lesssim \sum_{j=2}^{\infty} \left(\frac{t}{r_B}\right)^{n+2} j^{-2} \\ &\lesssim \left(\frac{t}{r_B}\right)^{n+2}. \end{aligned}$$

From (1) we now obtain by self-adjointness that for every ball $B \subset M$,

$$(4) \quad \|1_B T_{t^m} 1_{M \setminus 2B}\|_{2 \rightarrow 2} = \|1_{M \setminus 2B} T_{t^m} 1_B\|_{2 \rightarrow 2} \lesssim \left(\frac{t}{r_B}\right)^{n+2}, \quad t > 0.$$

Moreover, writing $1_{C_1(B)} = 1_{2B} - 1_B$, we see that

$$\begin{aligned} \|1_{C_1(B)} T_{t^m} 1_{M \setminus C_1^*(B)}\|_{2 \rightarrow 2} &= \|1_{2B \setminus B} T_{t^m} 1_{M \setminus 4B}\|_{2 \rightarrow 2} \\ &\leq \|1_{2B} T_{t^m} 1_{M \setminus 4B}\|_{2 \rightarrow 2} + \|1_B T_{t^m} 1_{M \setminus 4B}\|_{2 \rightarrow 2} \\ &\lesssim \left(\frac{t}{2r_B}\right)^{n+2} \end{aligned}$$

Finally, for any ball $B \subset M$ and any $k \geq 2$, we have

$$1_{C_k(B)} = 1_{2^k B} - 1_{2^{k-1} B}, \quad 1_{M \setminus C_k^*(B)} = 1_{M \setminus 2^{k+1} B} + 1_{2^{k-2} B}.$$

Therefore, by applying (4) with balls $2^k B$ and $2^{k-1} B$, and (2) with the ball $2^{k-2} B$, we obtain

$$\begin{aligned} \|1_{C_k(B)} T_{t^m} 1_{M \setminus C_k^*(B)}\|_{2 \rightarrow 2} &\leq \|1_{2^k B} T_{t^m} 1_{M \setminus 2(2^k B)}\|_{2 \rightarrow 2} + \|1_{2^{k-1} B} T_{t^m} 1_{M \setminus 2(2^k B)}\|_{2 \rightarrow 2} \\ &\quad + \|1_{C_2(2^{k-2} B)} T_{t^m} 1_{2^{k-2} B}\|_{2 \rightarrow 2} \\ &\lesssim \left(\frac{t}{2^k r_B}\right)^{n+2} + 2^{2n} \exp\left(-c\left(\frac{2^{k-2} r_B}{t}\right)^{\frac{m}{m-1}}\right), \quad t > 0, \end{aligned}$$

where by (3) we have

$$\exp\left(-c\left(\frac{2^{k-2} r_B}{t}\right)^{\frac{m}{m-1}}\right) \lesssim \left(\frac{t}{2^k r_B}\right)^{n+2},$$

as required. \square

The main result of this article relies on the following fact that a spectral gap implies exponential decay in time. Recall [20, Lemma 2.9]: if $(e^{-tL})_{t>0}$ satisfies $\text{GGE}_m(2, 2)$, then for any integer $j \geq 0$, also $((tL)^j e^{-tL})_{t>0}$ satisfies $\text{GGE}_m(2, 2)$.

Proposition 3. *If $(e^{-tL})_{t>0}$ satisfies $\text{GGE}_m(2, 2)$ and $\inf \sigma(L) > 0$, then, for small $\delta > 0$, also $(e^{-t(L-\delta)})_{t>0}$ satisfies $\text{GGE}_m(2, 2)$. In this case, for some $\delta > 0$ we have*

$$\|1_{E'} t^m L e^{-t^m L} 1_E\|_{2 \rightarrow 2} \lesssim e^{-\delta t^m}, \quad t > 0,$$

whenever $E, E' \subset M$. Moreover, for every ball $B \subset M$ and every $k \geq 0$ we have

$$\|1_{C_k(B)} t^m L e^{-t^m L} 1_{M \setminus C_k^*(B)}\|_{2 \rightarrow 2} \lesssim e^{-\delta t^m} \left(\frac{t}{2^k r_B}\right)^{n+2}, \quad t > 0.$$

Proof. The first claim follows by a straightforward generalization of [7, Proposition 2.2]: If F is a bounded analytic function on \mathbb{C}_+ and $|F(t)| \lesssim \exp(\delta t - \gamma t^{-\frac{1}{m-1}})$ for all $t > 0$, then $|F(t)| \lesssim \exp(-\gamma t^{-\frac{1}{m-1}})$ for all $t > 0$. The same proof applies with the modification $u(\zeta) = F((\gamma/\zeta)^{m-1})$. This can then be applied to $F(z) = \langle e^{-z(L-\delta)} f, g \rangle$ and $\gamma = d(E, E')^{\frac{m}{m-1}}$ for arbitrary $f, g \in \mathcal{L}^2$ whenever $0 < \delta < \inf \sigma(L)$.

For such δ we then have

$$\|1_{E'} t^m L e^{-t^m(L-\delta)} 1_E\|_{2 \rightarrow 2} \leq \|1_{E'} t^m (L - \delta) e^{-t^m(L-\delta)} 1_E\|_{2 \rightarrow 2} + \delta t^m \|1_{E'} e^{-t^m(L-\delta)} 1_E\|_{2 \rightarrow 2},$$

for all $t > 0$ whenever $E, E' \subset M$, and therefore

$$\|1_{E'} t^m L e^{-t^m L} 1_E\|_{2 \rightarrow 2} \lesssim (1 + \delta t^m) e^{-\delta t^m} \lesssim e^{-\delta t^m/2}, \quad t > 0,$$

by uniform boundedness. Moreover, Proposition 2 implies that for every ball $B \subset M$ and every $k \geq 0$ we have

$$\|1_{C_k(B)} t^m L e^{-t^m L} 1_{M \setminus C_k^*(B)}\|_{2 \rightarrow 2} \lesssim e^{-\delta t^m/2} \left(\frac{t}{2^k r_B}\right)^{n+2}, \quad t > 0,$$

as required. \square

3. MOLECULAR DECOMPOSITION

In this section we establish a molecular decomposition on the local Hardy space \mathfrak{h}_L^1 . For comparison, see [20, Subsection 4.3], [13, Chapter 5], [6, Subsection 7.1] and [12, Theorem 1.3].

Theorem 4. Assume that $(e^{-tL})_{t>0}$ satisfies $\text{GGE}_m(1, 2)$. Then every $f \in \mathfrak{h}_L^1 \cap R(L)$ admits a decomposition into N -molecules a_j for any integer $N \geq 1$ in the sense that

$$f = \sum_j \lambda_j a_j, \quad \text{where} \quad \sum_j |\lambda_j| \approx \|f\|_{\mathfrak{h}_L^1}.$$

Here the series converges in \mathcal{L}^2 .

Definition 5. Let $N \geq 1$ be an integer. A function $a \in \mathcal{L}^2$ is called an N -molecule associated with a ball $B \subset M$ if

$$\|1_{C_k(B)} a\|_2 \leq 2^{-k} \mu(2^k B)^{-1/2}, \quad k \geq 0,$$

and either $r_B \geq 1$, or $r_B \leq 2$ and a is cancellative in the following sense: there exists a $b \in D(L^N)$ such that $a = L^N b$ and for all $j = 0, 1, \dots, N$ we have

$$\|1_{C_k(B)} (r_B^m L)^j b\|_2 \leq r_B^{mN} 2^{-k} \mu(2^k B)^{-1/2}, \quad k \geq 0.$$

Lemma 6. For any integer $N \geq 1$ there exist constants c_0, c_1, \dots, c_{N+2} such that

$$f = c_{N+2} \int_0^1 (t^m L)^{N+2} e^{-2t^m L} f \frac{dt}{t} + c_{N+1} L^{N+1} e^{-2L} f + \dots + c_1 L e^{-2L} f + c_0 e^{-2L} f$$

for any $f \in R(L)$.

Proof. Fixing an integer $N \geq 1$ and an $f \in R(L)$ we start from the usual Calderón reproducing formula

$$f = c_{N+2} \left(\int_0^1 + \int_1^\infty \right) (t^m L)^{N+2} e^{-2t^m L} f \frac{dt}{t}$$

and show that

$$(5) \quad c_{N+2} \int_1^\infty (t^m L)^{N+2} e^{-2t^m L} f \frac{dt}{t} = c_{N+1} L^{N+1} e^{-2L} f + \dots + c_1 L e^{-2L} f + c_0 e^{-2L} f$$

for some constants c_0, c_1, \dots, c_{N+1} . Now, integrating by parts we see that for any $j \geq 1$ and for all $\lambda \geq 0$ we have

$$\begin{aligned} \int_1^\infty (t^m \lambda)^{j+1} e^{-2t^m \lambda} \frac{dt}{t} &= -\frac{1}{2} \int_1^\infty (t^m \lambda)^j \partial_t (e^{-2t^m \lambda}) dt \\ &= -\frac{1}{2} \left[(t^m \lambda)^j e^{-2t^m \lambda} \right]_{t=1}^\infty + \frac{1}{2} \int_1^\infty \partial_t ((t^m \lambda)^j) e^{-2t^m \lambda} dt \\ &= -\frac{1}{2} \lambda^j e^{-2\lambda} + \frac{mj}{2} \int_1^\infty (t^m \lambda)^j e^{-2t^m \lambda} \frac{dt}{t}. \end{aligned}$$

Iterating this from $j = N+1$ to $j = 0$ and using the spectral theorem we obtain (5). \square

Proof of Theorem 4. Let $f \in \mathfrak{h}_L^1 \cap R(L)$ and let $N \geq 1$ be an integer. We rewrite the reproducing formula as

$$f = \pi_1 u + \pi_2 f, \quad u(\cdot, t) = t^m L e^{-t^m L} f,$$

where

$$\pi_1 u = c_{N+2} \int_0^1 (t^m L)^{N+1} e^{-t^m L} u(\cdot, t) \frac{dt}{t}$$

and

$$\pi_2 f = c_{N+1} L^{N+1} e^{-2L} f + \dots + c_1 L e^{-2L} f + c_0 e^{-2L} f.$$

By atomic decomposition on local tent spaces (see [6, Theorem 3.6] or [1, Theorem 4.5]) we can write

$$u = \sum_j \lambda_j u_j$$

where u_j are *tent atoms* in the sense that each is supported in a box $B \times (0, r_B)$ and has

$$\left(\int_0^{r_B} \|u(\cdot, t)\|_2^2 \frac{dt}{t} \right)^{1/2} \leq \mu(B)^{-1/2}.$$

Claim: If u is a tent atom in $B \times (0, r_B)$, then $\pi_1 u$ is a constant multiple of a cancellative N -molecule associated with B . Choosing

$$b = \int_0^1 t^{m(N+1)} L e^{-t^m L} u(\cdot, t) \frac{dt}{t}$$

we have $L^N b = \pi_1 u$. Let $j = 0, 1, \dots, N$. For any g supported in $C_k(B)$, $k \geq 0$, with $\|g\|_2 \leq 1$ we have

$$\begin{aligned} |\langle (r_B^m L)^j b, g \rangle| &= \left| \int_M (r_B^m L)^j \int_0^1 t^{m(N+1)} L e^{-t^m L} u(\cdot, t) \frac{dt}{t} g d\mu \right| \\ &= \left| \int_0^{r_B \wedge 1} \int_B u(\cdot, t) (r_B^m L)^j t^{m(N+1)} L e^{-t^m L} g d\mu \frac{dt}{t} \right| \\ &\leq \left(\int_0^{r_B} \|u(\cdot, t)\|_2^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^{r_B \wedge 1} \|1_B (r_B^m L)^j t^{m(N+1)} L e^{-t^m L} g\|_2^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \mu(B)^{-1/2} r_B^{mj} \left(\int_0^{r_B \wedge 1} t^{2m(N-j)} \|1_B (t^m L)^{j+1} e^{-t^m L} g\|_2^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

For $k = 0, 1$ we simply estimate

$$\|1_B (t^m L)^{j+1} e^{-t^m L} g\|_2 \lesssim \|g\|_2 \leq 1,$$

while for $k \geq 2$ we see that, since $t \leq r_B$,

$$\|1_B (t^m L)^{j+1} e^{-t^m L} g\|_2 \lesssim 2^{nk} \exp\left(-c \left(\frac{2^k r_B}{t}\right)^{\frac{m}{m-1}}\right) \|g\|_2 \lesssim 2^{-\frac{n}{2}k} \exp(-c' 2^k).$$

Therefore for all $k \geq 0$ we have

$$\begin{aligned} \|1_{C_k(B)} (r_B^m L)^j b\|_2 &\lesssim \mu(B)^{-1/2} 2^{-\frac{n}{2}k} \exp(-c' 2^k) r_B^{mj} \left(\int_0^{r_B} t^{2m(N-j)-1} dt \right)^{1/2} \\ &\lesssim r_B^{mN} 2^{-k} \mu(2^k B)^{-1/2}, \end{aligned}$$

and the claim has been verified.

Turning to π_2 we cover M with a countable disjoint family \mathcal{Q} of sets Q each of which is contained in a ball B_Q of radius one. Then we may write

$$\pi_2 f = \sum_{Q \in \mathcal{Q}} \left(c_{N+1} L^{N+1} e^{-2L} (1_Q f) + \dots + c_1 L e^{-2L} (1_Q f) + c_0 e^{-2L} (1_Q f) \right),$$

where each $c_j L^j e^{-2L} (1_Q f)$ is a constant multiple of a noncancellative atom associated with the ball B_Q : since $(t L e^{-tL})_{t>0}$ satisfies $\text{GGE}_m(1, 2)$, it follows from Proposition 2 that for every $k \geq 0$,

$$\|1_{C_k(B_Q)} L^j e^{-2L} (1_Q f)\|_2 \lesssim \mu(B)^{-1/2} 2^{nk} \exp(-c 2^{\frac{m}{m-1}k}) \|f\|_1 \lesssim 2^{-k} \mu(2^k B)^{-1/2}.$$

This finishes the proof. \square

4. MAIN RESULT

Theorem 7. *Assume that $(e^{-tL})_{t>0}$ satisfies $\text{GGE}_m(1, 2)$. If $\inf \sigma(L) > 0$, then $\|Sf\|_1 \lesssim \|S_{loc}f\|_1 + \|f\|_1$, i.e. $\mathfrak{h}_L^1 = H_L^1$.*

Proof. The proof of the required estimate $\|Sf\|_1 \lesssim \|S_{loc}f\|_1 + \|f\|_1$ reduces to showing that $\|S_\infty a\|_1 \lesssim 1$ for all N -molecules a when N is large enough (here S_∞ is defined as S , but with \int_1^∞ replacing \int_0^∞). Indeed, an arbitrary $f \in \mathfrak{h}_L^1 \cap \mathcal{L}^2$ can be decomposed into N -molecules a_j for any $N \geq 1$ so that $f = \sum_j \lambda_j a_j$, and therefore

$$\|Sf\|_1 \leq \|S_{loc}f\|_1 + \sum_j |\lambda_j| \|S_\infty a_j\|_1.$$

Let a be an N -molecule associated with a ball $B \subset M$. If a is cancellative and $N > n/(2m)$, then $\|S_\infty a\|_1 \leq \|Sa\|_1 \lesssim 1$ by [20, Lemma 4.13] (see also [17, Corollary 3.6]). We may therefore assume that a is noncancellative and $r_B \geq 1$. Consider the following decomposition (cf. [2, Subsection 6.2]):

$$M \times (1, \infty) = \left(\bigcup_{k=0}^{\infty} C_k(B) \times (1, 2^k r_B) \right) \cup \left(\bigcup_{k=1}^{\infty} 2^{k-1} B \times (2^{k-1} r_B, 2^k r_B) \right).$$

Then, by Hölder's inequality,

$$\begin{aligned} \|S_\infty a\|_1 &\lesssim \sum_{k=0}^{\infty} \mu(2^k B)^{1/2} \left(\int_1^{2^k r_B} \|1_{C_k(B)} t^m L e^{-t^m L} a\|_2^2 \frac{dt}{t} \right)^{1/2} \\ &\quad + \sum_{k=1}^{\infty} \mu(2^k B)^{1/2} \left(\int_{2^{k-1} r_B}^{2^k r_B} \|1_{2^{k-1} B} t^m L e^{-t^m L} a\|_2^2 \frac{dt}{t} \right)^{1/2}, \end{aligned}$$

and it suffices to show that,

$$(6) \quad \left(\int_1^{\infty} \|1_{C_k(B)} t^m L e^{-t^m L} a\|_2^2 \frac{dt}{t} \right)^{1/2} \lesssim 2^{-k} \mu(2^k B)^{-1/2}, \quad k \geq 0,$$

and (since $r_B \geq 1$)

$$(7) \quad \left(\int_{2^{k-1}}^{\infty} \|1_{2^{k-1} B} t^m L e^{-t^m L} a\|_2^2 \frac{dt}{t} \right)^{1/2} \lesssim 2^{-k} \mu(2^k B)^{-1/2}, \quad k \geq 1.$$

We prove (6) for a fixed $k \geq 0$ by studying a in two pieces $1_{C_k^*(B)} a$ and $1_{M \setminus C_k^*(B)} a$. Firstly, by Proposition 3, for all $t > 0$ we have

$$\|1_{C_k(B)} t^m L e^{-t^m L} (1_{C_k^*(B)} a)\|_2 \lesssim e^{-\delta t} \|1_{C_k^*(B)} a\|_2,$$

and therefore

$$\begin{aligned} \left(\int_1^{\infty} \|1_{C_k(B)} t^m L e^{-t^m L} (1_{C_k^*(B)} a)\|_2^2 \frac{dt}{t} \right)^{1/2} &\lesssim \|1_{C_k^*(B)} a\|_2 \left(\int_1^{\infty} e^{-2\delta t} \frac{dt}{t} \right)^{1/2} \\ &\lesssim 2^{-k} \mu(2^k B)^{-1/2}. \end{aligned}$$

Secondly, by Proposition 3, for all $t > 0$ we have

$$\|1_{C_k(B)} t^m L e^{-t^m L} (1_{M \setminus C_k^*(B)} a)\|_2 \lesssim e^{-\delta t} \left(\frac{t}{2^k r_B} \right)^{n+2} \|a\|_2.$$

Therefore

$$\begin{aligned} \left(\int_1^\infty \|1_{C_k(B)} t^m L e^{-t^m L} (1_{M \setminus C_k^*(B)} a)\|_2^2 \frac{dt}{t} \right)^{1/2} &\lesssim (2^k r_B)^{-n-2} \|a\|_2 \left(\int_1^\infty e^{-2\delta t} t^{2n+3} dt \right)^{1/2} \\ &\lesssim 2^{-(n+2)k} \mu(B)^{-1/2} \\ &\lesssim 2^{-k} \mu(2^k B)^{-1/2} \end{aligned}$$

where in the second step we used the assumption that $r_B \geq 1$.

We then turn to (7) and, fixing a $k \geq 1$, divide a into $1_{2^k B} a$ and $1_{M \setminus 2^k B} a$. Again, by Proposition 3, for all $t > 0$ we have

$$\|1_{2^{k-1} B} t^m L e^{-t^m L} (1_{2^k B} a)\|_2 \lesssim e^{-\delta t} \|1_{2^k B} a\|_2,$$

where

$$\|1_{2^k B} a\|_2 \leq \sum_{l=0}^k \|1_{C_l(B)} a\|_2 \leq \sum_{l=0}^k 2^{-l} \mu(2^l B)^{-1/2} \lesssim 2^{nk/2} \mu(2^k B)^{-1/2}.$$

Therefore

$$\begin{aligned} &\left(\int_{2^{k-1}}^\infty \|1_{2^{k-1} B} t^m L e^{-t^m L} (1_{2^k B} a)\|_2^2 \frac{dt}{t} \right)^{1/2} \\ &\lesssim 2^{(n/2+1)k} \left(\int_{2^{k-1}}^\infty e^{-2\delta t} \frac{dt}{t} \right)^{1/2} 2^{-k} \mu(2^k B)^{-1/2}, \end{aligned}$$

where

$$\left(\int_{2^{k-1}}^\infty e^{-2\delta t} \frac{dt}{t} \right)^{1/2} \lesssim \left(\int_{2^{k-1}}^\infty \frac{dt}{t^{n+3}} \right)^{1/2} \approx 2^{-k(n/2+1)},$$

as required. Finally, noting that $2^{k-1} B = C_0(2^{k-1} B)$ and $M \setminus 2^k B = M \setminus C_0^*(2^{k-1} B)$, we apply Proposition 3 to see that for all $t > 0$ we have

$$\|1_{2^{k-1} B} t^m L e^{-t^m L} (1_{M \setminus 2^k B} a)\|_2 \lesssim e^{-\delta t} \left(\frac{t}{2^k r_B} \right)^{n+2} \|a\|_2$$

and therefore

$$\begin{aligned} &\left(\int_{2^{k-1}}^\infty \|1_{2^{k-1} B} t^m L e^{-t^m L} (1_{M \setminus 2^k B} a)\|_2^2 \frac{dt}{t} \right)^{1/2} \\ &\lesssim (2^k r_B)^{-n-2} \|a\|_2 \left(\int_1^\infty e^{-2\delta t} t^{2n+3} dt \right)^{1/2} \\ &\lesssim 2^{-k} \mu(2^k B)^{-1/2}, \end{aligned}$$

which concludes the proof. □

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